optical thickness of the medium increases over all limits, the solution for finite medium reduces to that derived by the author.

The scattering process in the radiative transfer problem, as formulated by the author and by the forementioned investigators, is defined only for the intensity of the scattered radiation and is given by a simple scalar function. However, for the cases of anisotropic scattering which occur in the planetary and stellar atmospheres, the scattering produces polarization, which substantially affects the intensity distribution, as demonstrated by Chandrasekhar. A more realistic solution of radiative transfer in a medium with anisotropic scattering requires the introduction of polarization effects, neglected in the forementioned studies. As shown by Chandrasekhar, the simple scalar equations have to be replaced by vectorial equations, the intensity or quantum emission by a matrix with four elements, the so-called Stokes parameters. In this case, it is difficult to use the probabilistic method, and it is preferable to solve the problem with the use of either the principles of invariance or the method of invariant imbedding (Bellman and Kalaba[6]). The resulting equations for the matrices defining the diffuse reflection and transmission have been derived recently by Sekera[7] and the method of their solution demonstrated for the case of Rayleigh scattering in a finite plane-parallel inhomogeneous medium.

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Theory of an Inertial System

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In this article, the possibility of obtaining a geographically oriented accelerometer platform with a natural oscillation period of 84.4 min is examined in the first approximation. The influence of gyroscope drift on the motion of a gyrostabilized platform and an accelerometer platform is studied, and errors in the determination of the coordinates of a moving object generated by drift of the "reference" coordinate system are derived. The study is conducted under the assumption that ideal elements are used in the system (with the exception of the floating gyroscopes), i.e., elements not containing inherent instrumental and design errors.

Independent of type and construction, the use of any inertial system is based on the determination of a course which is applied to automatic or semi-automatic control of a moving object. The self-contained determination of the instantaneous location and other elements of the motion of an object is accomplished in inertial systems by measuring acceleration, gravitation, angular velocity, and time in some previously chosen coordinate system.

1. The inertial system under examination has the following layout (Fig. 1).

A platform with two orthogonally placed accelerometers A_x and A_y for measuring acceleration along the x and y axes was installed on a three-axis gyrostabilized platform, which maintained constant orientation relative to the fixed stars, using an "equatorial support," the axis of which was directed along the axis of the earth's diurnal rotation[3] (Fig. 1).

The accelerometer platform was maintained in a geographic coordinate system by servosystems which included various computers, a clock, servomotors, and other elements.

Coriolis and centripetal accelerations are eliminated from the output signals of the accelerometers by special computers; the corrected output signals of the accelerometers go to the inputs of corresponding integrators.[2]

Let us examine the operation of the servocircuits. When an object's latitude φ undergoes accelerated change, a signal from accelerometer A_y enters the integrator I_{Iy}, the output signal of which is proportional to the object's velocity V_1. The correcting device CD_y converts the output signal of integrator I_{Iy} into a signal which is proportional to the angular velocity V_1/R, which is again integrated by the integrator I_{IIy} and applied to the amplifier U_y. The amplified signals enter the control winding of the servomotor SM_x, and the latter turns the accelerometer about the axis of suspension x.

When the object's longitude λ undergoes accelerated change, a signal from accelerometer A_x enters integrator I_{Ix}. The output signal of the correcting device CD_x, proportional to $V_2/R \cos\varphi$, enters the integrator I_{IIx}, is again integrated, and enters the summation device SD. Here it is added to a signal from the clock C, proportional to the angle of turning due to the earth's diurnal rotation. After amplification in the amplifier U_x, the signal u_{SD} enters the control winding of the servomotor SM_{zG}, which turns the accelerometer platform about the O_1z_G axis.

The sensitive elements of the gyrostabilized platform are three floating inertial gyroscopes (*FIG*) whose angular momenta are oriented perpendicular to the axes of stabilization. The output signals of the gyroscopes are distributed through a

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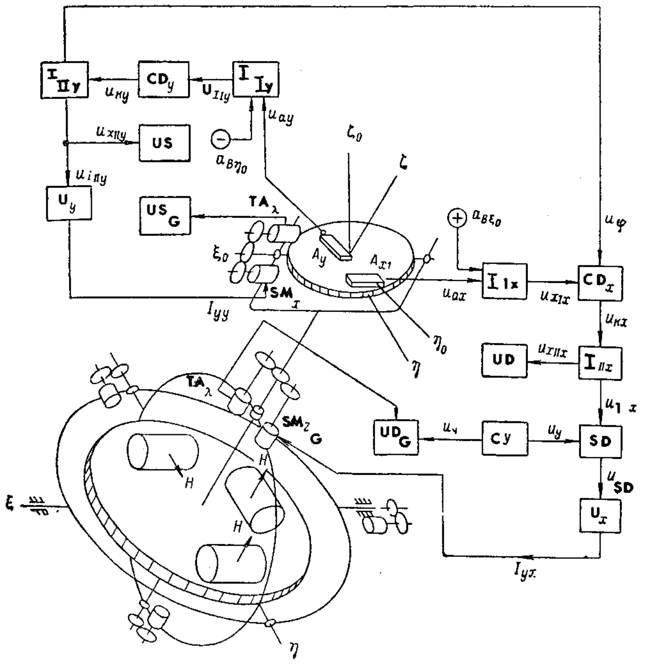

Fig. 1

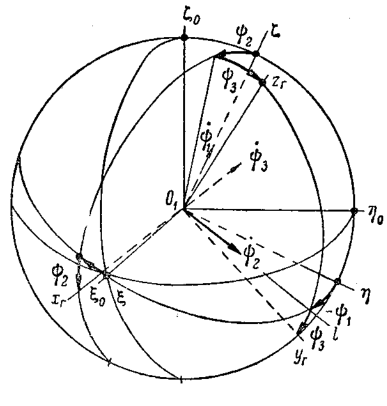

Fig. 2

coordinate converter between the amplifiers of corresponding servomotors of the platform.

The instantaneous values of latitude φ and longitude λ of the moving object can be obtained in the system by special angle pickoffs TA_φ and TA_λ connected to the corresponding axes.

2. The gyrostabilized platform of the inertial system is intended to maintain a "reference" coordinate system $\xi\eta\zeta$, which is fixed relative to the stars, from the following axes of orientation (Fig. 2):

ζ—directed upward along earth's axis of diurnal rotation
ξ—directed to east in initial time instant
η—forms right-hand coordinate system with ξ and ζ axes

Let us connect the coordinate system $x_Gy_Gz_G$ with the gyrostabilized platform. Its axes are arranged as follows when the gimbal angles $\psi_i = 0$ $(i = 1, 2, 3)$: z_G is directed upward along the platform's axis of rotation about the inner gimbal support; y_G coincides with the axis of rotation of the inner gimbal about the outer one; x_G coincides with the axis of rotation of the outer gimbal about the case of the instrument and forms a right-hand coordinate system with the y_G and x_G axes.

A table of the cosines of the angles between axes of the $\xi\eta\zeta$ and $x_Gy_Gz_G$ systems when the angles ψ_i are small can be obtained easily from Table 1,[2] substituting ψ_2, ψ_3, and ψ_1 for α, β, and δ, respectively.

We obtain expressions for the projections of the instantaneous angular velocity of the platform's turning about the x_G, y_G, z_G axes, using Table 1 and Figure 2:

$$\omega_{xG} = -\dot{\psi}_3 - \dot{\psi}_1\psi_2 \qquad \omega_{yG} = \dot{\psi}_2 - \dot{\psi}_1\psi_3$$
$$\omega_{zG} = \dot{\psi}_1 + \dot{\psi}_2\psi_3 \tag{1}$$

The precessional equations of motion of the gyrostabilized platform, without taking into account gimbal friction, can be set up using Fig. 1:

$$H(\omega_{yG} + p\chi_1) + k_d\cdot k_y\cdot u_{c1} = 0$$
$$H(\omega_{xG} - p\chi_2) - k_d\cdot k_y\cdot u_{c2} = 0$$
$$H(\omega_{zG} + p\chi_3) + k_d\cdot k_y\cdot u_{c3} = 0 \tag{2}$$

where H is the angular momentum of the gyroscope; χ_i the angles of deflection of the floating gyroelements from the zero position; u_{ci} the output voltages of the gyroscopes' angle pickoffs; and k_d and k_y the transmission coefficients of the servomotor and amplifier.

Henceforth, in order to simplify the differential equations of motion, we shall not take into account the gyroscopic torques which depend on the platform's angular velocity. These gyroscopic torques can be compensated if two electrically connected identical floating inertial gyroscopes with oppositely directed kinetic momenta are used for stabilization along each of the axes of the platform. In order to average the total output signal of each pair of gyroscopes, it is necessary to provide an electronic averaging device. Using paired gyroscopes also permits elimination of inertial errors due to angular acceleration about the axes of precession of the floating inertial gyroscopes.

The equations of motion of the floating gyroscopes, operating under conditions of geometric stabilization, without taking time constants and inertial errors into account, can, for linear angle pickoffs, be represented in the form

$$u_{c2} = \frac{k_cH}{pk_T}\left(\omega_{xG} + \frac{M_{B2}}{H}\right)$$
$$u_{c1} = \frac{k_cH}{pk_T}\left(\omega_{yG} - \frac{M_{B1}}{H}\right) \tag{3}$$
$$u_{c3} = \frac{k_cH}{pk_T}\left(\omega_{zG} - \frac{M_{B3}}{H}\right)$$

where k_T and k_c are the coefficients of damping and amplification of the angle pickoff, and M_{Bi} is the torque about the gyroscope output axes.

If the gyroscope has a low drift rate, the angle of deflection of the platform from the inertial coordinate system $\xi\eta\zeta$ after a long time interval will remain small. This assumption permits the components of the angular velocity due to kinematic coupling to be neglected.

With the foregoing assumptions, it is not difficult to obtain the precessional equations of motion of the gyrostabilized platform from Eqs. (2), (3), and (1) (neglecting second-order terms):

$$\psi_1 = \frac{1}{H}\int_0^t M_{B3}(\tau)d\tau$$
$$\psi_2 = \frac{1}{H}\int_0^t M_{B1}(\tau)d\tau \tag{4}$$
$$\psi_3 = \frac{1}{H}\int_0^t M_{B2}(\tau)d\tau$$

The gyrostabilized platform in time leaves the inertial coordinate system owing to the presence of disturbance torques about the axes of precession of the floating inertial gyroscopes. From Eqs. (4) one can obtain a concrete expression for the angles ψ_i, if the torques M_{Bi} are given.

3. As already indicated, the accelerometer platform is used in the system for making a model of the geographic coordinate system $\xi_0\eta_0\zeta_0$, the axes of which are directed to the east, the north, and the zenith, respectively.

In order to study the motion of the platform, let us connect an xyz system to it, whose axes are directed as follows (Fig. 3): z is perpendicular to the platform's plane upward; x is along platform's axis of rotation in its gimbals directed to the east; and y lies in the plane of the platform and forms a right-hand coordinate system with the x and z axes.

Since the accelerometer platform has two degrees of freedom relative to the gyrostabilized platform, two differential equations are required to define the dynamics of the servocircuits which accomplish stabilization of the platform in the geographic coordinate system.

In order to set up the equations of motion of the platform, let us use the torque equation

$$I_k\dot{\omega}_k = M_k \qquad k = x, z_G$$

which can be represented in the form

$$\int_0^t \omega_k(\tau)d\tau = \frac{1}{I_k}\int_0^t\int_0^{t'} M_k(\tau)d\tau d\tau' \tag{5}$$

From the diagram of the system (Fig. 1), it is apparent that, if the output signals of the second integrators enter the control windings of the servomotors, then an appropriate choice of parameters of the individual elements of the accelerometer platform will cause it to behave similarly to an undamped pendulum with $T = 84.4$ min.

The differential equations of motion of the accelerometer platform are easy to obtain using Eq. (5) and Fig. 1; for brevity, let us write them in symbolic operator form

$$\frac{1}{p}\,\omega_x = -\frac{1}{I_x}\,W_d(p)W_y(p)W_i^2(p)W_{ky}(p)W_a(p)a_y(p)$$

$$\frac{1}{p}\,\omega_{zG} = \frac{1}{I_{zG}}\,W_d(p)W_y(p)W_{c3}(p)\,[W_r(p)\omega_e + W_i^2(p)W_{kx}(p)W_a(p)a_x(p)] \tag{6}$$

where $W_d(p)$, $W_y(p)$, $W_i(p)$, $W_a(p)$, $W_{kx}(p)$, $W_{ky}(p)$, $W_r(p)$, and $W_{c3}(p)$ are the transmission functions of the servomotor, amplifier, integrator, accelerometer, correcter, clock, and summation device, respectively, and $a_x(p)$ and $a_y(p)$ are the corrected accelerations along the x and y axes of the platform.

The instantaneous values of the misalignment angles of the accelerometer platform in the $\xi_0\eta_0\zeta_0$ coordinate system (Fig. 3) can be represented in the form

$$\alpha = \alpha_0 + \frac{\Delta\omega_{\eta_0}}{p} \qquad \beta = \beta_0 + \frac{\Delta\omega_x}{p}$$

$$\delta = \delta_0 + \frac{\Delta\omega_{\zeta_0}}{p} \tag{7}$$

where α_0, β_0, δ_0 are the initial misalignment angles of the platform in the $\xi_0\eta_0\zeta_0$ coordinate system; and $\Delta\omega_x$, $\Delta\omega_{\eta 0}$, $\Delta\omega_{\zeta 0}$ the angular velocities of the accelerometer platform about the corresponding axes due to the gyrostabilized platform's drift.

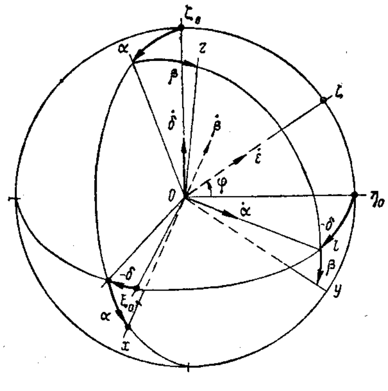

Fig. 3

Table 1

	ξ	η	ζ
x_G	1	ψ_1	$-\psi_2$
y_G	$-\psi_1$	1	$-\psi_3$
z_G	ψ_2	ψ_3	1

If it is assumed that coriolis and centripetal accelerations are subtracted from the output signals of the accelerometers, the expressions for the accelerations in the case of low vehicle velocity can be represented by the following approximation:

$$a_x = pV_2 + pV_1\delta - g\alpha$$

$$a_y = pV_1 - pV_2\delta - g\beta \tag{8}$$

The accelerometer platform turns relative to the fixed stars about the x and z_G axes with the angular velocities

$$\omega_x = \left(p\beta + \frac{V_1}{R} + \Delta\omega_x\right)$$

$$\omega_{zG} = p\epsilon + \omega_e + \frac{V_2}{R\cos\varphi} + \Delta\omega_{zG} \tag{9}$$

where ϵ is the misalignment angle of the accelerometer platform about the Oz_G axis.

Assuming that ideal elements are used in the servocircuits, after substituting the angular velocities from Eqs. (9) and the acceleration from Eqs. (8) into Eqs. (6), we obtain

$$p\beta + \frac{V_1}{R} + \Delta\omega_x = \frac{k_dk_yk_i^2\,k_kk_a}{I_x}\left(V_1 - V_2\delta - g\frac{\beta}{p}\right)$$

$$p\epsilon + \omega_e + \frac{V_2}{R\cos\varphi} + \Delta\omega_{zG} = \frac{k_dk_yk_{c3}}{I_{zG}}\left[k_r\omega_e + k_i^2k_kk_a\,\frac{1}{\cos\varphi}\left(V_2 + V_1\delta - g\frac{\alpha}{p}\right)\right] \tag{10}$$

where k_d, k_y, k_{c3}, k_k, k_i, k_a denote the transmission functions of the corresponding devices.

Imposing on the system the following conditions:

$$\frac{k_dk_yk_i^2k_kk_a}{I_x} = \frac{1}{R} \qquad \frac{k_{dy}k_{c3}}{I_{zG}} = 1$$

$$k_i^2k_kk_a = \frac{1}{R} \qquad k_r = 1 \tag{11}$$

we obtain from Eqs. (10):

$$p\beta + \frac{g}{R}\frac{\beta}{p} + \frac{V_2}{R}\,\delta = -\Delta\omega_x$$

$$p\epsilon + \frac{g}{R}\,\frac{\alpha}{p\cos\varphi} - \frac{V_1}{R\cos\varphi}\,\delta = -\Delta\omega_{zG} \tag{12}$$

From Fig. 3, when the angles α, β, and δ are small, it follows that

$$p\alpha = p\epsilon\cos\varphi \qquad p\delta = p\epsilon\sin\varphi \tag{13}$$

Table 2

	ξ	η	ζ
ξ_0	$-\sin\lambda^*$	$\cos\lambda^*$	0
τ_0	$-\sin\varphi\cos\lambda^*$	$-\sin\varphi\sin\lambda^*$	$\cos\varphi$
ζ_0	$\cos\varphi\cos\lambda^*$	$\cos\varphi\sin\lambda^*$	$\sin\varphi$

Multiplying the second equation by $\cos\varphi$ and $\sin\varphi$, respectively, we obtain differential equations which describe the behavior of the accelerometer platform

$$p\alpha + \nu^2 \frac{\alpha}{p} - \frac{V_1}{R}\delta = -\Delta\omega_{\eta_0}$$

$$p\beta + \nu^2 \frac{\beta}{p} + \frac{V_2}{R}\delta = -\Delta\omega_x$$

$$p\delta + \nu^2 \frac{\delta}{p} - \frac{V_1}{R}\delta \tan\varphi = -\Delta\omega_{\zeta_0} \quad (14)$$

where $\nu^2 = g/R$.

As is apparent from Eqs. (14), the presence of the components

$$\frac{V_1}{R}\delta \qquad \frac{V_2}{R}\delta \qquad \frac{V_1}{R}\delta\tan\varphi$$

does not allow free oscillations of the accelerometer platform at the Schuler period. However, if the platform is somehow corrected using a precision gyroscopic compass, these components will, when the angle δ is small, be values of the second order of smallness and can be neglected. In this case, Eqs. (14) takes the form

$$\dot\alpha + \nu^2 \int_0^t \alpha d\tau = -\Delta\omega_{\eta_0}$$

$$\dot\beta + \nu^2 \int_0^t \beta d\tau = -\Delta\omega_x$$

$$\dot\delta + \nu^2 \int_0^t \delta d\tau = -\Delta\omega_{\zeta_0} \quad (15)$$

In order to determine the components of the angular velocity vector of the damping of the gyrostabilized platform along the axes of the geographic coordinate system, let us set up a table of cosines of the angles between the axes of the $\xi\eta\zeta$ and $\xi_0\eta_0\zeta_0$ systems (Table 2).

In Table 2, λ^* should be understood as

$$\lambda^* = \lambda + \omega_e t$$

Using Table 2 and Fig. 2, it is easy to obtain

$$\Delta\omega_{\xi_0} = \dot\psi \cos\lambda^* + \dot\psi_3 \sin\lambda^*$$

$$\Delta\omega_{\eta_0} = \dot\psi_1 \cos\varphi + (\dot\psi_3 \cos\lambda^* - \dot\psi_2 \sin\lambda^*)\sin\varphi$$

$$\Delta\omega_{\zeta_0} = \dot\psi_1 \sin\varphi + (\dot\psi_2 \sin\lambda^* - \dot\psi_3 \cos\lambda^*)\cos\varphi \quad (16)$$

Since the vehicle moves with a low velocity, differentiating Eq. (15), having substituted the angular velocities of drift from Eq. (16) and neglecting values of the second order of smallness, we obtain

$$\ddot\alpha + \nu^2\alpha = (\ddot\psi_2 \sin\lambda^* - \ddot\psi_3 \cos\lambda^*)\sin\varphi - \ddot\psi_1 \cos\varphi$$

$$\ddot\beta + \nu^2\beta = -(\ddot\psi_2 \cos\lambda^* + \ddot\psi_3 \sin\lambda^*)$$

$$\ddot\delta + \nu^2\delta = (\ddot\psi_3 \cos\lambda^* - \ddot\psi_2 \sin\lambda^*)\cos\varphi - \ddot\psi_1 \sin\varphi \quad (17)$$

In real instruments, gyroscopes with drift rates having random time functions are used. However, in order to study the influence of drift of the gyrostabilized platform upon the accelerometer platform's motion and upon the errors in the output signals of the system, we shall assume that in some individual case, after a small time interval, the gyro drift torques vary as follows:

$$M_{B1} = a_1 t + a_2 \qquad M_{B2} = b_1 t + b_2$$

$$M_{B3} = c_1 t + c_2 \quad (18)$$

The differential equations of motion of the accelerometer platform, after substituting the torque M_{Bi} into (17) and taking Eqs. (4) into account, will have the form

$$\ddot\alpha + \nu^2\alpha = \left[\frac{a_1}{H}\sin(\omega_e t + \lambda) - \frac{b_1}{H}\cos(\omega_e t + \lambda)\right]\sin\varphi - \frac{c_1}{H}\cos\varphi$$

$$\ddot\beta + \nu^2\beta = -\left[\frac{a_1}{H}\cos(\omega_e t + \lambda) + \frac{b_1}{H}\sin(\omega_e t + \lambda)\right]$$

$$\ddot\delta + \nu^2\delta = \left[\frac{b_1}{H}\cos(\omega_e t + \lambda) - \frac{a_1}{H}\sin(\omega_e t - \lambda)\right]\cos\varphi - \frac{c_1}{H}\sin\varphi \quad (19)$$

Assuming that at the initial time instant $t = 0$:

$$\alpha = \alpha_0 \qquad \beta = \beta_0 \qquad \delta = \delta_0$$

$$\dot\alpha = -\Delta\omega_{\eta_0}(0) \qquad \dot\beta = -\Delta\omega_{\xi_0}(0)$$

$$\dot\delta = -\Delta\omega_{\zeta_0}(0) \quad (20)$$

since the angles of deflection of the xyz coordinate system from the geographic coordinate system $\xi_0\eta_0\zeta_0$ are small, we obtain, after integrating (19), the equations of motion of the accelerometer platform:

$$\alpha = \alpha_0^* \cos\nu t + \frac{\dot\alpha_0 d^*}{\nu H}\sin\nu t + \frac{L}{\nu^2 - \omega_e^2}\sin\varphi\sin(\omega_e t + \lambda + \rho_\alpha) - \frac{c_1}{\nu^2 H}\cos\varphi$$

$$\beta = \beta_0^* \cos\nu t + \frac{\beta_{0d}^*}{\nu H}\sin\nu t + \frac{L}{\nu^2 - \omega_e^2}\sin(\omega_e t + \lambda + \rho_\beta)$$

$$\delta = \delta_0^* \cos\nu t + \frac{\dot\delta_{0d}^*}{\nu H}\sin\nu t + \frac{L}{\nu^2 - \omega_e^2}\cos\varphi\cdot\sin(\omega_e t + \lambda + \rho_\delta) - \frac{c_1}{\nu^2 H}\sin\varphi \quad (21)$$

where

$$\alpha_0^* = \alpha_0 + \frac{(b_1\cos\lambda_0 - a_1\sin\lambda_0)}{H(\nu^2 - \omega_e^2)}\sin\varphi_0 + \frac{c_1}{\nu^2 H}\cos\varphi_0$$

$$\dot\alpha_{0d}^* = \left[(a_2\sin\lambda_0 - b_2\cos\lambda_0) - \frac{\omega_e(a_1\cos\lambda_0 + b_1\sin\lambda_0)}{\nu^2 - \omega_e^2}\right]\sin\varphi_0 - c_2\cos\varphi_0$$

$$B_0^* = \beta_0 + \frac{a_1\cos\lambda_0 + b_1\sin\lambda_0}{H(\nu^2 - \omega_e^2)}$$

$$\beta_{0d}^* = -(a_2\cos\lambda_0 + b_2\sin\lambda_0) + \frac{\omega_e(b_1\cos\lambda_0 - a_1\sin\lambda_0)}{\nu^2 - \omega_e^2}$$

$$\delta_0^* = \delta_0 + \frac{(a_1\sin\lambda_0 - b_1\cos\lambda_0)}{H(\nu^2 - \omega_e^2)}\cos\varphi + \frac{c_1}{\nu^2 H}\sin\varphi_0$$

$$\dot\delta_{0d}^* = (b_2\cos\lambda_0 - a_2\sin\lambda_0)\cos\varphi_0 + \left[\frac{\omega_e(a_1\cos\lambda_0 + b_1\sin\lambda_0)}{\nu^2 - \omega_e^2} - c_2\right]\sin\varphi_0$$

$$L = \frac{\sqrt{a_1^2 + b_1^2}}{H}$$

$$\rho_\alpha = -\arctan(b_1/a_1) \qquad \rho_\beta = \arctan(a_1/b_1)$$

$$\rho_\delta = -\arctan(b_1/a_1)$$

As follows from Eqs. (21), the motion of the accelerometer platform for a particular gyro drift torque function M_{Bi} is accomplished with two periods: $T = 84.4$ min and $T_e = 24$ hr.

4. Errors in the determination of the instantaneous coordinates φ and λ of the vehicle, taken in the form of output signals of the second integrators, are represented in the form

$$\Delta\varphi = -\nu^2 \int_0^t \int_0^{t'} \beta d\tau d\tau'$$

$$\Delta\lambda = -\nu^2 \sec\varphi' \int_0^t \int_0^{t'} \alpha d\tau d\tau' \tag{22}$$

since

$$\Delta\varphi = \varphi_c(t) - \varphi(t) \qquad \Delta\lambda = \lambda_c(t) - \lambda(t) \tag{23}$$

where $\varphi(t)$ and $\lambda(t)$ are the true coordinates of the object, and $\varphi_c(t)$ and $\lambda_c(t)$ are the coordinates at the output of the second integrators.

Let us assume that φ and λ are constants, in view of the comparatively low velocity of the vehicle. Then, substituting α and β from Eqs. (21) and assuming that $t = 0$, $S = V = 0$, after double integration (without taking the small term ω_e^2/ν^2 into account), we obtain

$$\Delta\varphi = \frac{c_1 \cos\varphi}{H} t^2 - \left[\frac{\dot{\alpha}_{0d}^*}{H} + \frac{L}{\omega_e}\sin\varphi\cos(\lambda + \rho_\alpha)\right] t + \alpha_0^*(\cos\nu t - 1) + \frac{\alpha_{0d}^*}{\nu H}\sin\nu t + \frac{\mathrm{L}}{\omega_e^2}\sin\varphi\,[\sin\omega_e t + (\lambda + \rho_\alpha) - \sin(\lambda + \rho_\alpha)]$$

$$\Delta\lambda = \sec\varphi\left\{\left[\frac{\cos(\lambda + \rho_\beta)}{\omega_e} - \frac{\dot{\beta}_{0d}^*}{H}\right] t + \dot{\beta}_0^*(\cos\nu t - 1) + \frac{\dot{\beta}_{0d}^*}{\nu H}\sin\nu t - \frac{L}{\omega_e^2}[\sin(\omega_e t + \lambda + \rho_\beta) - \sin(\lambda + \rho_\beta)]\right\} \tag{24}$$

If the disturbing torques cause only a constant drift of the gyroscopes, errors in the determination of φ and λ are obtained from Eqs. (24), assuming $a_1 = b_1 = c_1 = 0$:

$$\Delta\varphi = \frac{1}{H}[c_2\cos\varphi_0 - (a_2\sin\lambda_0 - b_2\cos\lambda_0)] \times \left(t - \frac{1}{\nu}\sin\nu t\right) + \alpha_0(\cos\nu t - 1)$$

$$\Delta\lambda = \sec\varphi\left\{\frac{1}{H}(a_2\cos\lambda_0 + b_2\sin\lambda_0) \times \left(t - \frac{1}{\nu}\sin\nu t\right) + \beta_0(\cos\nu t - 1)\right\} \tag{25}$$

As can easily be seen, even in this most simple case, errors in the determination of the coordinates grow with time, which causes additional errors in the system; for example, coriolis and centripetal accelerations will be calculated with some error.

5. Now let us examine errors in the determination of the latitude and longitude of a moving object when they are taken as angles between the corresponding axes of two platforms.

From Fig. 2 it is apparent that the latitude of the vehicle may be obtained by means of a special pickoff TA_Φ between the y and z_G axes. Owing to the gyroscope drift the z_G axis of the gyrostabilized platform is at some time instantly deflected from its primary orientation at an angle

$$\Phi_y = \int_0^t \Delta\omega\xi_0 d\tau = \frac{1}{\omega_e}\left[Lt\sin(\omega_e t + \lambda + \rho_\alpha) + \frac{L}{\omega_e}\sin(\omega_e t + \lambda + \rho_\beta)\right] + L_1\sin(\omega_e t + \lambda + \rho_c) \tag{26}$$

where

$$L_1 = \frac{1}{H}\sqrt{a_2^2 + b_2^2}$$

$$\rho_c = -\operatorname{arc\,tan}(b_2/a_2)$$

Therefore, the error in the determination of latitude, neglecting values of the second order of smallness, will be the following:

$$\Delta\varphi_G = \beta_0^*\cos\nu t + \frac{\dot{\beta}_{0d}^*}{\nu H}\sin\nu t + \frac{1}{\omega_e}\left\{Lt\sin(\omega_e t + \lambda + \rho_\alpha) + \frac{L}{\omega_e}[\sin(\omega_e t + \lambda + \rho_\beta) - \sin(\lambda + \rho_\beta)] + L_1\sin(\omega_e t + \lambda + \rho_c)\right\} \tag{27}$$

and for constant drift

$$\Delta\varphi_G = \beta_0\cos\nu t - (a_2\cos\lambda_0 + b_2\sin\lambda_0)\sin\nu t + \frac{L_1}{\omega_e}\sin(\omega_e t + \lambda + \rho_c) \tag{28}$$

There is a special angle pickoff TA_λ in the system for determining the object's longitude geometrically. It measures the angle between the x_G and x axes:

$$\Phi = \omega_e t + \lambda + \epsilon + \varphi_\lambda$$

However, in order to obtain λ, it is necessary to eliminate the angle $\omega_e t$ mechanically or electrically. Then the pickoff's output signal will be proportional to the angle

$$\lambda + \epsilon + \varphi_\lambda$$

and, therefore, the error in determining longitude will be

$$\Delta\lambda_G = \epsilon + \varphi_\lambda \tag{29}$$

The error in the determination of λ due to drift of the gyrostabilized platform is determined easily from Eqs. (4) and (18):

$$\Phi_\lambda = (c_1 t^2 + c_2 t) + \psi_{10} \tag{30}$$

The accelerometer platform's motion in the plane of the angles ϵ is found using the second equation in (12):

$$\dot{\epsilon} + \nu^2 \int_0^t \epsilon d\tau = -\dot{\psi}_1$$

Differentiating the equation with respect to time, we obtain

$$\ddot{\xi} + \nu^2\epsilon = -\frac{c_1}{H} \tag{31}$$

Assuming that at the initial time instant $t = 0$, $\epsilon = \epsilon_0$, $\dot{\epsilon} = -\dot{\psi}_1^{(0)}$, we find the equations of motion of the platform in the plane of the angles

$$\epsilon = \left(\epsilon_0 + \frac{c_1}{\nu^2 H}\right)\cos\nu t - \frac{c_2}{\nu H}\sin\nu t - \frac{c_1}{\nu^2 H} \tag{32}$$

Thus, the error in the geometric determination of the longitude of the vehicle has the following form:

$$\Delta\lambda_G = \frac{1}{H}(c_1t^2 + c_2t) + \left(\epsilon_0 + \frac{c_1}{\nu^2 H}\right)\cos\nu t - \frac{c_2}{\nu H}\sin\nu t + \psi_{10} - \frac{c_1}{\nu^2 H} \quad (33)$$

and for constant drift

$$\Delta\lambda_G = \frac{c_2}{H}t + \epsilon_0 \cos\nu t - \frac{c_2}{\nu H}\sin\nu t + \psi_{10} \quad (34)$$

If follows from the foregoing equations that errors in the determination of latitude and longitude using this particular method of instrumentation will be less when the gyroscope drift is constant.

In conclusion, it should be noted that this system can work without additional correction only during a comparatively short time interval. Therefore, for precise navigation it is necessary to have additional information sources, the use of which would provide periodic correction for the gyrostabilized platform.

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Reviewer's Comment

The paper presents an analysis of an inertial system using a five-gimbal platform: a two-gimbal platform containing two orthogonal horizontal accelerometers mounted on the inner element of a three-gimbal platform which also contains three orthogonal gyros. The element containing the gyros is stabilized so that it is fixed in inertial space, with the outer gimbal axis of the accelerometer platform free to rotate about an axis parallel to the earth's axis of rotation.

A comparison is made between the errors in position output caused by gyro drift using two methods of instrumentation: double integration of the outputs of the accelerometers and measurement of the gimbal angles of the accelerometer platform. The equations are simplified for the case in which the vehicle velocity is slow compared with the velocity due to the earth's rate of rotation and indicate the effects of the Schuler oscillation and of the periodicity due to the earth's rate of rotation.

The author's main conclusion is that the system errors using the platform pickoff method of instrumentation are inherently less than those using double integration of acceleration, considering constant gyro drifts as the only source errors. This conclusion is, in this reviewer's opinion, wrong, based as it is on a series of errors:

1) In Table 2, and thereafter, $\sin\lambda^*$ should be replaced by $-\cos\lambda^*$, $\cos\lambda^*$ by $\sin\lambda^*$; similar substitutions should be made for λ.

2) In Eqs. (17), terms of the form $\dot{\psi}\omega_e$ are omitted; it is precisely these terms which cancel the apparently important system errors of Eqs. (25).

3) In Eqs. (24) and (25), the position errors causing errors in latitude and longitude are interchanged; i.e., the right-hand side of the first equation and the contents of the outer bracket in the right-hand side of the second equation should be interchanged.

Despite these errors, there are features of this paper which offer an insight into the state of development of inertial systems in the Soviet Union. One of these is the proposition of a five-gimbal platform. Such a platform offers definite simplifications in computation but at the expense of platform size and weight. The availability of small, lightweight computers would make such a platform obsolete, except, perhaps, for naval vessels.

Another interesting point is the form of the gyro drift torques given in Eqs. (18); it is apparently made up of two parts: a constant drift torque and a drift torque changing at a constant rate, sometimes called "trend." The inclusion of this latter term in a simplified analysis, in which accelerometer errors and random gyro drifts are not considered, would indicate that it is considered very important. It is perhaps also worth noting that the periodic drift rates due to the change of the gravity vector with respect to the space-stabilized gyro platform at the earth's rate of rotation, caused by mass unbalances in the gyros, are not considered.

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